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SOME CHARACTERISATIONS OF THE ELLIPSOID

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ABSTRACT

We show that a convex body K of dimension $d \geqq 3$ is an ellipsoid if it has any of the following properties: (1) the "grazes" of all points close to K are flat, (2) all sections of small diameter are centrally symmetric, (3) parallel $(d-1)$-sections close to the boundary are width-equivalent, (4) K is strictly convex and all $(d-1)$-sections close to the boundary are centrally symmetric; the last two results are deduced from their 3-dimensional cases which were proved by Aitchison.

1. Introduction

If $K \subset E^d$ is a convex body and $p \in E^d \backslash K$, the *graze of* p *with respect to* K is the set of points of K contained in support lines passing through p. Our main result is:

THEOREM 1. *Let K be a convex body in E^d $(d \geqq 3)$ such that for some $\delta > 0$, for every $p \in E^d \backslash K$ whose distance from K is less than δ, the graze of p is contained in a hyperplane. Then K is an ellipsoid.*

A similar result was proved in 3 dimensions, under assumptions of smoothness and with $\delta = \infty$, by Kubota [7].

THEOREM 2. *Let K be a convex body in E^d $(d \geqq 3)$, $2 \leqq j \leqq d-1$, and suppose that for some $\delta > 0$, every j-dimensional section of K having diameter less than δ is centrally symmetric. Then K is an ellipsoid.*

Theorems 3 and 4 are generalisations to higher dimensions of results proved in 3 dimensions by Aitchison [1, 2], and are deduced from these results by simple induction arguments; the obvious approach in terms of sections of sections

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appears to fail, and Aitchison (private communication) has indicated that he is no longer convinced by his induction argument in [1] which was intended to prove Theorem 3 of the present paper.

We first give some definitions. Let $C \subset E^d$ be a compact convex set. We define the *support function* h_C of C by $h_C(\boldsymbol{u}) = \sup\{\boldsymbol{x}.\boldsymbol{u} : \boldsymbol{x} \in C\}$, and $H_C(\alpha, \boldsymbol{u}) = \{\boldsymbol{x} \in E^d : \boldsymbol{x}.\boldsymbol{u} = h_C(\boldsymbol{u}) - \alpha\}$ where $\boldsymbol{u}$ is a unit vector and $\alpha > 0$. The *width* $w_C(\boldsymbol{u})$ *of* C *in direction* $\boldsymbol{u}$ is $h_C(\boldsymbol{u}) + h_C(-\boldsymbol{u})$, and is non-negative and translation invariant; two compact convex sets C, D are *width-equivalent* if there is a positive constant λ such that $w_C(\boldsymbol{u}) = \lambda w_D(\boldsymbol{u})$ for all unit vectors $\boldsymbol{u}$. Homothetic compact convex sets are width-equivalent, and convex bodies of constant width are width-equivalent. If C and D are compact convex sets in E^{d-1} and f is an orthogonal embedding of E^{d-1} in E^d, then C and D are width-equivalent if and only if $f(C)$ and $f(D)$ are width-equivalent.

THEOREM 3. *Let K be a convex body in E^d $(d \geqq 3)$ and suppose there exists $\varepsilon > 0$ such that for every unit vector $\boldsymbol{u}$ and $0 < \alpha < \beta < \varepsilon$ the sections $H_K(\alpha, \boldsymbol{u}) \cap K$ and $H_K(\beta, \boldsymbol{u}) \cap K$ are width-equivalent. Then K is an ellipsoid.*

An interesting extension of this result in 3 dimensions is given in [2].

THEOREM 4. *Let K be a strictly convex body in E^d $(d \geqq 3)$ and suppose that for each unit vector $\boldsymbol{u}$ there exists $\varepsilon(\boldsymbol{u}) > 0$ such that $H_K(\alpha, \boldsymbol{u}) \cap K$ is centrally symmetric for $0 < \alpha < \varepsilon(\boldsymbol{u})$. Then K is an ellipsoid.*

The assumption of strict convexity is essential, as may be seen by considering the sum of a ball with a line segment.

2. Preliminary results

LEMMA 1. *Let C be a 2-dimensional convex body which is smooth and strictly convex, let U be an arc of ∂C and let $\boldsymbol{b}$ be an inner point of C. For each $\boldsymbol{a} \in \partial C$, let $\boldsymbol{a}'$ be the other point of ∂C having a parallel support line. If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}'$ are collinear for each $\boldsymbol{a} \in U$, then there is a positive number t such that $\boldsymbol{a} - \boldsymbol{b} = t(\boldsymbol{b} - \boldsymbol{a}')$ for all $\boldsymbol{a} \in U$.*

PROOF. Choose polar coordinates with centre $\boldsymbol{b}$, so that $\partial C = \{(r(\theta), \theta) : \theta$ real$\}$, where $r(\theta) = r(\theta + 2\pi)$ for each θ, and $U = \{(r(\theta), \theta) : \alpha \leqq \theta \leqq \beta\}$, say. Then for $\boldsymbol{a} = (r(\theta), \theta) \in U$, we can write $\boldsymbol{a}' = (r'(\theta), \theta + \pi)$. For any $\boldsymbol{a} = (r(\theta), \theta) \in \partial C$ let $k(\theta)$ be the angle from $\boldsymbol{a} - \boldsymbol{b}$ to the support line to C at $\boldsymbol{a}$, measured in the negative sense (see Fig. 1, which shows the case $\boldsymbol{a} \in U$).

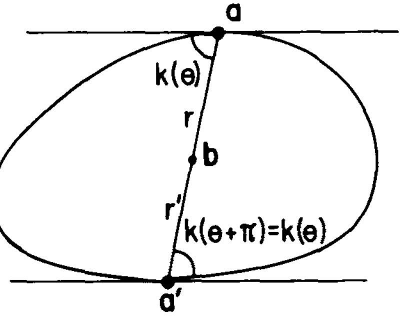

Fig. 1

Then $dr/d\theta = -r(\theta)\cot k(\theta)$, so for $\alpha \leqq \theta \leqq \beta$ we have

$$\frac{d}{d\theta}\left(\frac{r}{r'}\right)=\frac{r'\dfrac{dr}{d\theta}-r\dfrac{dr'}{d\theta}}{(r')^2}=\frac{-r'r\cot k(\theta)+rr'\cot k(\theta+\pi)}{(r')^2}=0$$

so r/r' is constant, which proves the Lemma.

If K is a convex body in E^d and $\boldsymbol{p} \in E^d \backslash K$, the affine hull of the graze of p is called the *graze flat* of $\boldsymbol{p}$, and has dimension $d-1$ or d.

LEMMA 2. *Let C be a strictly convex body in E^2 and $0<\delta<1$. Suppose that each line segment I with $C \cap \operatorname{aff} I = \varnothing$ and such that each point of I lies within distance δ of C has the property that the graze lines of its points are concurrent at an inner point of C. Then C is an ellipse.*

PROOF. We first show that C is smooth. Suppose this is false, and let $\boldsymbol{x}$ be a non-smooth point of C. Since the non-smooth points of C are countable, we may choose a sequence $\{\boldsymbol{x}(i)\}$ of smooth points, with respective support lines $l(i)$, such that $\boldsymbol{x}(i)\rightarrow \boldsymbol{x}$. Let l be a support line to C at $\boldsymbol{x}$. Then $\boldsymbol{y}(i)=l\cap l(i)\rightarrow \boldsymbol{x}$; choose i^* such that $|\boldsymbol{y}(i^*)-\boldsymbol{x}|<\delta$. We can then choose a line segment I containing $\boldsymbol{y}(i^*)$, such that $C\cap \operatorname{aff} I=\varnothing$, $l(i^*)\neq \operatorname{aff} I$, and all points of I have distance less that δ from $\boldsymbol{x}$ and lie on support lines containing $\boldsymbol{x}$. Then the points of I have graze lines which are concurrent at $\boldsymbol{x}$ but at no other point, contradicting the hypothesis of the Lemma. We conclude that C is smooth.

For any point $\boldsymbol{y}\in\partial C$, let l_y denote the support line of C at $\boldsymbol{y}$; if q is any non-parallel line disjoint from C, let $q(\boldsymbol{y})$ be the point of ∂C (distinct from $\boldsymbol{y}$) whose support line passes through $q\cap l_y$. We can choose $\eta>0$ so that for any two points $\boldsymbol{a}$, $\boldsymbol{b}$ on ∂C with $|\boldsymbol{a}-\boldsymbol{b}|<\eta$, l_a intersects l_b at a point with distance less than δ from C; for such a pair $\boldsymbol{a}$, $\boldsymbol{b}$ we define $\mathscr{A}(\boldsymbol{a},\boldsymbol{b})$ to be that component of $\partial C\backslash\{\boldsymbol{a},\boldsymbol{b}\}$ for which l_c intersects l_a for all $\boldsymbol{c}$ in $\mathscr{A}(\boldsymbol{a},\boldsymbol{b})$.

For distinct points $\boldsymbol{a}, \boldsymbol{b} \in \partial C$ whose distance apart is less than η, choose points $\boldsymbol{c}, \boldsymbol{d}$ in ∂C having distance less than η from $\boldsymbol{a}$, such that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{c}$ are distinct and lie in that order on $\mathrm{cl}\,\mathscr{A}(\boldsymbol{a}, \boldsymbol{c})$. Let $\boldsymbol{h} = l_a \cap l_c$, $\boldsymbol{j} = l_a \cap l_d$, $\boldsymbol{k} = l_b \cap l_c$ and $m = \mathrm{aff}\{\boldsymbol{j}, \boldsymbol{k}\}$. For any line p through $\boldsymbol{j}$ which meets relint $[\boldsymbol{h}, \boldsymbol{k}]$, define $f_p(\boldsymbol{x}) = p(m(\boldsymbol{x}))$ for $\boldsymbol{x} \in \mathscr{A}(\boldsymbol{a}, \boldsymbol{b})$ (see Fig. 2). Then f_p maps $\mathscr{A}(\boldsymbol{a}, \boldsymbol{b})$ bijectively onto $\mathscr{A}(\boldsymbol{a}, \boldsymbol{b}')$, where $\boldsymbol{b}'$ is a point of $\mathscr{A}(\boldsymbol{a}, \boldsymbol{b})$ which may be chosen arbitrarily by appropriate choice of p. Notice that f_p is order-preserving, continuous, and extends continuously by $f_p(\boldsymbol{a}) = \boldsymbol{a}$, $f_p(\boldsymbol{b}) = \boldsymbol{b}'$. For $\boldsymbol{x} \in \mathscr{A}(\boldsymbol{a}, \boldsymbol{b})$, the chords $[\boldsymbol{x}, m(\boldsymbol{x})]$ are concurrent at s (say) on $[\boldsymbol{a}, \boldsymbol{d}]$, and the chords $[m(\boldsymbol{x}), f_p(\boldsymbol{x})]$ are concurrent at $\boldsymbol{t}_p$ (say) on $[\boldsymbol{a}, \boldsymbol{d}]$, for fixed p.

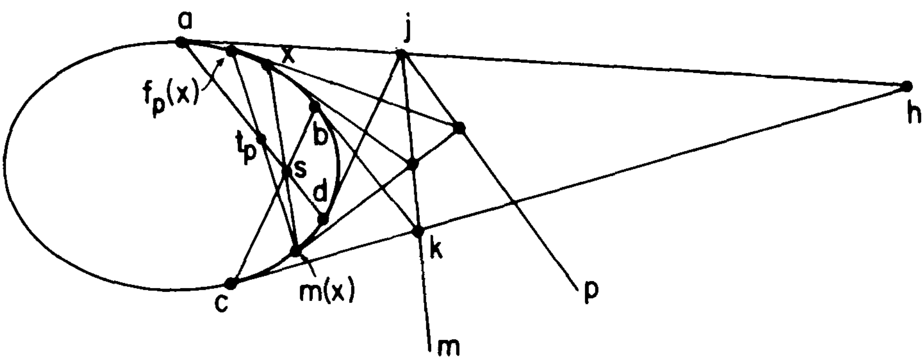

Fig. 2

We may apply a projective transformation Φ to map m to infinity and ensure that $[\boldsymbol{a}, \boldsymbol{d}]$ is perpendicular to l_a (we shall use the same symbols as above to denote images under Φ; see Fig. 3). Observe that p, l_a, l_d are all parallel, that l_c is parallel to l_b and that l_x is parallel to $l_{m(x)}$. We choose coordinates so that $\boldsymbol{a} = 0$, $\boldsymbol{d} = (d_1, 0)$ with $d_1 > 0$, and l_a is the x_2-axis with $b_2 > 0$. Then the lines p are those lines $\{\boldsymbol{x} : x_1 = -\beta\}$ with $\beta > 0$. For real λ write π_λ for the projective transformation

$$\pi_\lambda(x_1, x_2) = \left(\frac{x_1}{1 + \lambda x_1}, \frac{x_2}{1 + \lambda x_1}\right)$$

and let $\mathscr{F}$ be the group of projective transformations which have the form

$$\boldsymbol{x} \mapsto \left(\frac{\alpha x_1 + \xi}{\zeta x_1 + \sigma}, \frac{\gamma x_2}{\zeta x_1 + \sigma}\right)$$

and are non-singular. Then $\mathscr{F}$ contains all the maps π_λ and $\boldsymbol{x} \mapsto \boldsymbol{u} + \varepsilon(\boldsymbol{x} - \boldsymbol{u})$, with ε, $\boldsymbol{u}$ constants such that $\varepsilon \neq 0$, $u_2 = 0$, and their inverses.

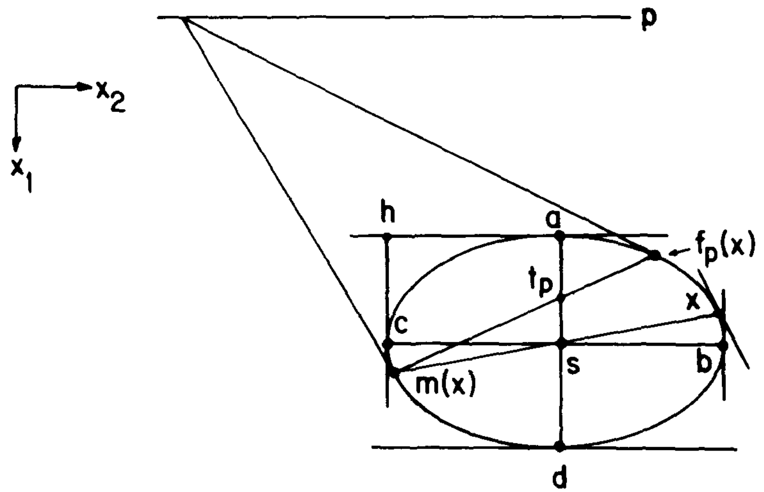

Fig. 3

For $x \in \mathcal{A}(a, b)$, l_x is parallel to $l_{m(x)}$ and the chords $[x, m(x)]$ are concurrent at s. Hence by Lemma 1 there is a constant $\nu \neq 0$ such that $m(x) - s = \nu(s - x)$ for all $x \in \mathcal{A}(a, b)$. For $y \in m\mathcal{A}(a, b)$ the lines l_y and $l_{p(y)}$ are concurrent on $p = \{x : x_1 = -\lambda^{-1}\}$, say, so the lines $\pi_\lambda l_y$ and $\pi_\lambda l_{p(y)}$ are parallel. Further the chords $[y, p(y)]$ are concurrent at t_p, so by Lemma 1 there is a constant μ such that $\pi_\lambda p(y) - \pi_\lambda t_p = \mu(\pi_\lambda t_p - \pi_\lambda(y))$ for $y \in m\mathcal{A}(a, b)$. Hence $f_p(x) = \pi_\lambda^{-1}(\pi_\lambda t_p + \mu(\pi_\lambda t_p - \pi_\lambda(s + \nu(s - x)))$ which shows that $f_p \in \mathcal{F}$.

Inverting Φ, we see that f_p is a projective transformation admissible for $\mathrm{cl}\,\mathcal{A}(a, b)$. By appropriate choice of p, we can ensure that any given point x of $\mathcal{A}(a, b)$ is mapped by f_p to any given point of $\mathcal{A}(a, x)$. It follows that for any point $u \in \partial C$ and $v \in \partial C$ with distance less than η from u, there is a non-singular projective transformation f and a neighbourhood U of u such that $f(u) = v$ and $f(U \cap \partial C) \subset \partial C$. Since ∂C is twice differentiable almost everywhere (see for example [4]), this implies that ∂C is twice differentiable everywhere. Since C is strictly convex, almost all, and hence all, points of ∂C have non-zero second derivatives (working in some coordinate system in the tangent and normal).

Let us re-apply Φ and return to the situation of Fig. 3. We can choose a sub-arc G of $\mathcal{A}(a, b)$ with a as one end, the other end being e say, such that the points of G can be parametrised by their x_2-coordinates. Then, by a form of Taylor's Theorem given in [6] (but note a misprint in some editions) G is an arc of a curve $x_1 = \varphi x_2^2 + o(x_2^2)$ for some $\varphi > 0$. For positive integers n let $p(n) = \{x : x_1 = 1/n\}$, and $f^n = f_{p(n)}$. As $n \to \infty$, $p(n)$ approaches l_a so $f^n(x) \to a$ for each $x \in G$. Since

$f^n(\boldsymbol{a}) = \boldsymbol{a}$, and f^n maps points of G to points whose coordinates have unchanged signs, we may write

$$f^n(\boldsymbol{x}) = \left(\frac{\alpha_n x_1}{\zeta_n x_1 + \beta_n}, \frac{\gamma_n x_2}{\zeta_n x_1 + \beta_n}\right)$$

where α_n, β_n and γ_n are non-zero and have the same sign.

Let Γ be the parabola $\{\boldsymbol{x}: x_1 = \varphi x_2^2\}$, and let $g(\boldsymbol{x}) = (x_1, \sqrt{x_1/\varphi}) \in \Gamma$ for $\boldsymbol{x} \in G$. Then $g(\boldsymbol{x})_2 = \sqrt{(x_2^2 + o(x_2^2))}$ and hence $g(\boldsymbol{x})_2/x_2 \to 1$ as $\boldsymbol{x} \to \boldsymbol{a}$. For fixed $\boldsymbol{x} \in G$ we have

$$\frac{g(f^n(\boldsymbol{x}))_2}{f^n(\boldsymbol{x})_2} \to 1$$

as $n \to \infty$. Now

$$\frac{[(f^n)^{-1}(g(f^n(\boldsymbol{x})))]_2}{[(f^n)^{-1}(f^n(\boldsymbol{x}))]_2} = \frac{g(f^n(\boldsymbol{x}))_2}{f^n(\boldsymbol{x})_2}$$

since f^n has the effect of a linear transformation on lines parallel to l_a; consequently, $[(f^n)^{-1}(g(f^n(\boldsymbol{x})))]_2 \to x_2$ as $n \to \infty$. Further, $[(f^n)^{-1}(g(f^n(\boldsymbol{x})))]_1 = x_1$.

Let $\Gamma_n = (f^n)^{-1}\Gamma$, so $(f^n)^{-1}(g(f^n(\boldsymbol{x}))) \in \Gamma_n$ for all $\boldsymbol{x} \in G$. The above discussion shows that for $0 < u < e_1$, we can choose a number $v_n(u)$ such that $(u, v_n(u)) \in \Gamma_n$ and $v_n(x_1) \to x_2$ as $n \to \infty$ for each $\boldsymbol{x} \in G$.

If $\zeta_n = 0$ for all sufficiently large n, then we may suppose for large n that $\Gamma_n = \{\boldsymbol{x}: \sigma_n x_1 = x_2^2\}$ where $\sigma_n > 0$, and so $v_n^2(u) = \sigma_n u$. It follows that σ_n tends to a limit $\sigma > 0$ as $n \to \infty$, so G is an arc of the parabola $\{\boldsymbol{x}: x_1 = \sigma x_2^2\}$.

If $\zeta_n \neq 0$ for infinitely many n, by choosing a subsequence, relabelling and rrrechoosing the constants, we may suppose that $\zeta_n = 1$ and α_n has the same sign for all n. Thus $\Gamma_n = \{\boldsymbol{y}: \sigma_n y_1(\beta_n + y_1) = y_2^2\}$ where $\sigma_n = \alpha_n/(\varphi\gamma_n^2)$ has the sign of α_n. We first consider the case $\sigma_n > 0$ for all n. Then

$$\Gamma_n = \{(u, v): (u + q_n)^2/q_n^2 - v^2/r_n^2 = 1\}$$

where $q_n = \beta_n/2$ and $r_n = \beta_n\sqrt{\sigma_n}/2$, so that

$$v_n^2(u) = \frac{r_n^2}{q_n}\left(2u + \frac{u^2}{q_n}\right) = \left(\frac{r_n}{q_n}\right)^2 (2uq_n + u^2).$$

If $\{q_n\}_{n=1}^\infty$ is unbounded, then $\{r_n^2/q_n\}_{n=1}^\infty$ must have a convergent subsequence, in which case G must be a line segment, contradicting the strict convexity of C. It is not possible that $q_n \to 0$, for then $\{r_n^2/q_n^2\}_{n=1}^\infty$ must converge in which case G is again a line segment. Thus some subsequence of $\{q_n\}_{n=1}^\infty$ has a positive limit q, and the corresponding subsequence of $\{r_n\}_{n=1}^{\infty}$ has a positive limit r. Then G is an arc of the hyperbola

$$\left\{(u, v)\colon \left(\frac{u+q}{q}\right)^2 - \left(\frac{v}{r}\right)^2 = 1\right\}.$$

We now suppose that $\sigma_n < 0$ for all n. Then

$$\Gamma_n = \left\{(u, v)\colon \left(\frac{u-q_n}{q_n}\right)^2 + \left(\frac{v}{r_n}\right)^2 = 1\right\}$$

for some positive q_n and r_n, so

$$v_n^2(u) = \frac{r_n^2}{q_n}\left(2u - \frac{u^2}{q_n}\right).$$

As before, $\{q_n\}_{n=1}^{\infty}$ is bounded, and if $q_n \to 0$ then $v_n^2(u)$ is negative for large n, which is impossible. We conclude that G is an arc of the ellipse

$$\left\{(u, v)\colon \left(\frac{u-q}{q}\right)^2 + \left(\frac{v}{r}\right)^2 = 1\right\}$$

for some real q and r. Thus, in all possible cases, G is an arc of a second-order curve.

Let us invert Φ. Let Λ be a maximal open arc of a second-order curve in ∂C. If Λ is not the whole of ∂C, then Λ has an end point z say. Let w be a point of Λ having distance less than η from z. Then there is a neighbourhood U of w and a non-singular projective transformation f admissible for U such that $f(w) = z$ and $f(U \cap \partial C) \subset \partial C$. Since $f(U \cap \Lambda)$ is an open subset of a second order curve and intersects Λ, we have a contradiction to the maximality of Λ. Hence ∂C is a second-order curve, and must therefore be an ellipse.

3. Proofs of the theorems

Proof of Theorem 1. First consider the case $d = 3$. We prove that K is strictly convex. Suppose this is false, and let I be a line segment in ∂K. Choose a point $x \in (\operatorname{aff} I)\backslash K$ having distance less than δ from K. The graze flat of x is a plane π containing I; let l be a line in π through x disjoint from K. There is a support plane ω through l distinct from π. Then $\omega \cap K$ is a non-empty subset of the graze of x but is not contained in π. This contradiction shows that K is strictly convex.

Let p be a fixed interior point of K, and let π be an arbitrary plane containing p. Consider any line segment $I \subset \pi$ such that $(\operatorname{aff} I) \cap (\pi \cap K) = \emptyset$ and every point of I has distance less than δ from $\pi \cap K$. There are just two support planes ω_1, ω_2 of K which contain I. Let q_1, q_2 be their respective points of contact with K. The line segment $[q_1, q_2]$ intersects $\pi \cap K$ at a relatively interior point q, since K is strictly convex. If $x \in I$, then q_1 and q_2 lie in the graze plane φ of x with respect to K. Thus $\varphi \cap \pi$ is a line through q, and the points of $(\varphi \cap \pi) \cap (\partial K \cap \pi)$ lie on support lines of $\pi \cap K$ which contain x, so $\pi \cap \varphi$ is the graze line of x with respect to $\pi \cap K$, relative to π. We have now shown that $\pi \cap K$ satisfies the hypothesis of Lemma 2, and is therefore an ellipse. Thus all sections of K through p are ellipses, which shows (see [3, p. 91]) that K is an ellipsoid.

We now suppose that $d \geqq 4$, let p be a fixed interior point of K and let Π be an arbitrary hyperplane containing p. If x is any point of $\Pi \backslash K$ whose distance from $\Pi \cap K$ is less than δ, the graze Δ of x with respect to K affinely generates a hyperplane Γ. We can choose a $(d-2)$-flat λ in Π through x but disjoint from K. Then some hyperplane Ω which contains λ but is distinct from Π supports K, which shows that $\Pi \neq \Gamma$. The graze of x with respect to $\Pi \cap K$ is $\Pi \cap \Delta$, which lies in the $(d-2)$-flat $\Pi \cap \Gamma$. Thus $\Pi \cap K$ satisfies the conditions of the Theorem in dimension $d-1$. If we make the inductive hypothesis that the Theorem holds in dimension $d-1$, then every $(d-1)$-dimensional section of K through p is an ellipsoid, so K is an ellipsoid (see [3, p. 91]). By induction, the Theorem is proved.

Before proving Theorem 2, we need a result of S.P. Olovjanischnikoff [8], which we will state as Lemma 3. If C is a convex body and u is a member of the unit sphere S^2, in E^3, let $Q(u) = \{\varepsilon > 0\colon H_C(\varepsilon', u) \cap C$ is non-empty and centrally symmetric for $0 < \varepsilon' < \varepsilon\}$. Define $\varepsilon(u) = \sup Q(u)$ if $Q(u) \neq \emptyset$, or $\varepsilon(u) = 0$ if $Q(u) = \emptyset$, and $N_C(u) = \{x \in \partial C\colon 0 \leqq h_C(u) - x \,.\, u < \varepsilon(u)\}$ or $N_C(u) = \emptyset$ if $\varepsilon(u) = 0$. The *face* $f_C(u)$ *of* C *in direction* u *is* $H_C(0, u) \cap C$.

LEMMA 3. *Let C be a convex body in E^3, A an open connected non-empty subset of ∂C, and D an open non-empty subset of S^2, such that $f_C(u) \subset A \subset N_C(u)$ for all $u \in D$. Then A is a subset of a second-order surface (that is, a paraboloid, ellipsoid or hyperboloid of two sheets) or of an elliptical cone whose apex is contained in A.*

PROOF OF THEOREM 2. First consider the case $d = 3$, $j = 2$. Let a be an extreme point of K, and let F be the set of points of K having distance at least $\delta/2$ from a. Then $\operatorname{conv} F$ is compact and $a \notin \operatorname{conv} F$, so we may strictly separate a from F with a plane: that is, there is a $v \in S^2$ and a real number α such that $a \,.\, v > \alpha > x \,.\, v$ for all $x \in F$. Let $\beta = (a \,.\, v - \alpha)/3$, and let $A = \{x \in \partial K\colon x \,.\, v > \alpha + 2\beta\}$, so that A is an open connected set in ∂K and $a \in A$. An easy contradiction argument proves the existence of an open set $D_1 \subset S^2$ with $\boldsymbol{v} \in D_1$ and $f_K(\boldsymbol{u}) \subset A$ for all $\boldsymbol{u} \in D_1$. We then choose an open set $D_2 \subset S^2$ with $\boldsymbol{v} \in D_2 \subset D_1$ and $\boldsymbol{x}.\boldsymbol{u} \geqq \alpha + \beta$ for all $\boldsymbol{u} \in D_2$ and $\boldsymbol{x} \in A$. Next we choose an open set $D_3 \subset S^2$ such that $\boldsymbol{v} \in D_3 \subset D_2$ and $\boldsymbol{x}.\boldsymbol{u} \leqq \alpha$ for all $\boldsymbol{u} \in D_3$ and $\boldsymbol{x} \in F$. For $\boldsymbol{u} \in D_3$, if $0 < \varepsilon < h_K(\boldsymbol{u}) - \alpha$, then $H_K(\varepsilon, \boldsymbol{u})$ does not intersect F, so that $H_K(\varepsilon, u) \cap K$ has diameter less than δ and is therefore centrally symmetric. Thus $f_K(\boldsymbol{u}) \subset A \subset N_K(\boldsymbol{u})$ for all $\boldsymbol{u} \in D_3$. By Lemma 3, A is a subset of a second-order surface or of an elliptical cone with apex $\boldsymbol{a}$.

In particular, this shows that K has no facets, since any facet would contain an extreme point of K. Suppose that B is a maximal open connected subset of an elliptical cone surface $\hat{B}$ such that $B \subset \partial K$ and the apex of $\hat{B}$ lies in B. We may assume that $\boldsymbol{o}$ is the apex of $\hat{B}$ and that $\hat{B} \cap S^2 \subset B$. Then for any $\boldsymbol{x}$ in $E^3 \backslash \{\boldsymbol{o}\}$, let $\text{ray}(\boldsymbol{x})$ be the set $\{\lambda \boldsymbol{x}: \lambda > 0\}$. For any $\boldsymbol{x} \in B \cap S^2$, $\text{ray}(\boldsymbol{x})$ intersects ∂K in a line segment $[\boldsymbol{o}, \boldsymbol{c}(\boldsymbol{x})]$ say; from this it follows that for all such $\boldsymbol{x}$, $\text{ray}(\boldsymbol{x})$ intersects B in a half-open line segment $[\boldsymbol{o}, \boldsymbol{g}(\boldsymbol{x}))$, and hence $\text{ray}(\boldsymbol{x})$ intersects $\text{cl}\, B$ in a line segment $[\boldsymbol{o}, \boldsymbol{b}(\boldsymbol{x})]$, where $\boldsymbol{b}(\boldsymbol{x})$ is a boundary point of B. Let $\boldsymbol{x} \in B \cap S^2$. If $\boldsymbol{b}(\boldsymbol{x})$ were an extreme point, then there would be an open connected subset A of an elliptical cone apex $\boldsymbol{b}(\boldsymbol{x})$ or of a second-order surface, with $\boldsymbol{b}(\boldsymbol{x}) \in A \subset \partial K$ which is impossible, since A would have to be a subset of $\hat{B}$, contradicting the maximality of B. We conclude that $\boldsymbol{b}(\boldsymbol{x})$ is relatively interior to a line segment $I \subset \partial K$, and I must be a subset of $\text{ray}(\boldsymbol{x})$, for otherwise $\text{aff}(I \cup [\boldsymbol{o}, \boldsymbol{b}(\boldsymbol{x})])$ would intersect K in a facet. This shows that $|\boldsymbol{c}(\boldsymbol{x})| > |\boldsymbol{b}(\boldsymbol{x})|$. Let $T = \text{cl}\, \boldsymbol{b}(B \cap S^2)$, so that T is a subset of the boundary of B, and hence any point of T is relatively interior to $[\boldsymbol{o}, \boldsymbol{c}(\boldsymbol{x})]$ for some $\boldsymbol{x}$. For each positive integer n let $T_n = \{y \in T: (1 + (n|y|)^{-1})y \in K\}$, so T_n is closed. Since T is a complete metric space and $T = \bigcup_{n=1}^{\infty} T_n$, by the Baire Category Theorem we can choose a point $\boldsymbol{r} \in T$, a positive integer n and a real number γ such that $0 < 4\gamma < 1/n$ and all points of T with distance less than 4γ from $\boldsymbol{r}$ lie in T_n. By the definition of T we can choose $\boldsymbol{x} \in B \cap S^2$ such that $|\boldsymbol{b}(\boldsymbol{x}) - \boldsymbol{r}| < \gamma$. Then $\boldsymbol{b}(\boldsymbol{x}) + \gamma \boldsymbol{x}$ belongs to $\partial K \backslash \text{cl}\, B$ so we can choose a real number μ such that $0 < \mu < \gamma$ and every point having distance less than μ from $\boldsymbol{b}(\boldsymbol{x}) + \gamma \boldsymbol{x}$ is not a member of $\text{cl}\, B$. Let Γ be an open arc of $B \cap S^2$ such that $\boldsymbol{x} \in \Gamma$ and $\text{ray}(\boldsymbol{y})$ contains a point having distance less than μ from $\boldsymbol{b}(\boldsymbol{x}) + \gamma \boldsymbol{x}$ for all $\boldsymbol{y} \in \Gamma$. Thus for $\boldsymbol{y} \in \Gamma$ we have $|\boldsymbol{b}(\boldsymbol{y})| < |\boldsymbol{b}(\boldsymbol{x})| + \gamma + \mu$. Since $\boldsymbol{b}(\boldsymbol{x}) \in \text{cl}\, B$, we can choose an open arc $\Gamma' \subset \Gamma$ such that for each $\boldsymbol{y} \in \Gamma'$, $\text{ray}(\boldsymbol{y})$ contains a point of B having distance less than γ from $\boldsymbol{b}(\boldsymbol{x})$, so that $|\boldsymbol{b}(\boldsymbol{y})| > |\boldsymbol{b}(\boldsymbol{x})| - \gamma$. Hence for $\boldsymbol{y}, \boldsymbol{y}' \in \Gamma'$ we have $||\boldsymbol{b}(\boldsymbol{y})| - |\boldsymbol{b}(\boldsymbol{y}')|| < 3\gamma$ and, by construction, for $\boldsymbol{y} \in \Gamma'$, $\boldsymbol{b}(\boldsymbol{y})$ lies on a line segment $[\boldsymbol{p}, \boldsymbol{q}]$ where $\boldsymbol{p}$ has distance less than μ from $\boldsymbol{b}(\boldsymbol{x}) + \gamma \boldsymbol{x}$ and $\boldsymbol{q}$ has distance less than γ from $\boldsymbol{b}(\boldsymbol{x})$, and $\boldsymbol{b}(\boldsymbol{x})$

has distance less than γ from $\boldsymbol{r}$, so that $\boldsymbol{b}(\boldsymbol{y})$ has distance less than 3γ from $\boldsymbol{r}$, which ensures that $|\boldsymbol{c}(\boldsymbol{y})|>|\boldsymbol{b}(\boldsymbol{y})|+1/n$. Hence $\{\lambda \boldsymbol{z}: \boldsymbol{z} \in \Gamma', 0<\lambda<|\boldsymbol{b}(\boldsymbol{y})|+\gamma\}$ is an open subset of $\hat{B}$ which contains $\boldsymbol{b}(\boldsymbol{y})$, for any $\boldsymbol{y} \in \Gamma'$, and which is contained in ∂K, contradicting the maximality of B. We conclude that every extreme point of K is contained in an open subset of a second-order surface in ∂K, and so, since any edge contains an extreme point, that K is strictly convex. Let G be a maximal open subset of a second order surface in ∂K. If $G \neq \partial K$, then G has a boundary point $\boldsymbol{a}$ which must be extreme, so there is a subset A of a second-order surface such that A is open and $\boldsymbol{a} \in A \subset \partial K$. Then A and G have an open subset in common, and so are subsets of the same surface, contradicting the maximality of G. Hence $G=\partial K$, so that K is an ellipsoid. This completes the case $d=3$, $j=2$.

We now suppose $n \geqq 3$ and that the result holds for $d=n$, $j=n-1$. Let $K \subset E^{n+1}$ satisfy the hypothesis of the Theorem for $d=n+1$, $j=n$. Consider an orthogonal projection Ω on an n-dimensional linear flat π such that no line segment in ∂K is parallel to $\pi^{\perp}$. For any $(n-1)$-flat ω in π, we have $\Omega((\omega+\pi^{\perp}) \cap K)=\omega \cap \Omega(K)$. For some $\varepsilon>0$, for each $(n-1)$-flat $\omega \subset \pi$ such that $\operatorname{diam}(\omega \cap \Omega(K))<\varepsilon$ we have $\operatorname{diam}((\omega+\pi^{\perp}) \cap K)<\delta$; otherwise, by taking $\varepsilon=1, \frac{1}{2}, \frac{1}{3}, \cdots$ we can prove the existence of a line segment of length δ parallel to $\pi^{\perp}$ in ∂K, contrary to the choice of π. Hence, since a projection of a centrally symmetric set is centrally symmetric, $\Omega(K)$ satisfies the hypothesis of the Theorem for $d=n$, $j=n-1$, so that $\Omega(K)$ is an ellipsoid. The set of unit vectors representing directions of line segments in ∂K has σ-finite $(n-1)$-measure on the unit sphere in E^{n+1} (see Ewald, Larman and Rogers [5]). Therefore, by taking limits, all n-dimensional orthogonal projections of K are ellipsoids, which ensures that K is an ellipsoid. (This may be deduced by polar duality from the result that for $k>l \geqq 2$, a k-dimensional convex body, all of whose l-dimensional sections through a fixed inner point are ellipsoids, is an ellipsoid, which is given by Busemann in [3, p. 91].) This inductive step completes the proof in the case $d \geqq 3$, $j=d-1$.

Finally we consider the case $d-2 \geqq j \geqq 2$. If W is a $(j+1)$-dimensional section of K, then every j-dimensional section of W having diameter less than δ is centrally symmetric, so that W is an ellipsoid by the cases already established. Thus all $(j+1)$-dimensional sections of K are ellipsoids, so K is an ellipsoid (see [3, p. 91]).

Proofs of Theorems 3 and 4. These results have been proved for $d=3$ by Aitchison [1, 2]. Let $d \geqq 4$ and suppose that Theorems 3 and 4 hold in $d-1$

dimensions. If K is a convex body in E^d and Ω is an orthogonal projection on a linear $(d-1)$-flat π, then for any unit vector $\boldsymbol{u} \in \pi$, we have $h_{\Omega K}(\boldsymbol{u}) = h_K(\boldsymbol{u})$ and $H_{\Omega K}(\alpha, \boldsymbol{u}) \cap \Omega K = \Omega(H_K(\alpha, \boldsymbol{u}) \cap K)$ for $\alpha > 0$. Using the fact that width-equivalence, central symmetry and strict convexity are inherited by orthogonal projections, we see that if K satisfies the conditions of Theorems 3 or 4, then ΩK satisfies the conditions of Theorems 3 or 4 respectively in $d-1$ dimensions. Hence all the $(d-1)$-dimensional orthogonal projections of K are ellipsoids, so K is an ellipsoid (this follows by duality from a result on p. 91 of [3]). This induction argument completes the proofs of Theorems 3 and 4.

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